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Special solutions from the variable separation approach: the Davey–Stewartson equation

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Abstract. A variable separation procedure for the Davey–Stewartson (DS) equation is proposed by using a prior ansatz to its bilinear form. The reduced equations for two variable separated fields have the same trilinear form although they possess different independent variables. The trilinear equation can be changed to a spacetime symmetric form and can be solved by means of a Boussinesq-type equation system. Whenever a pair of solutions of the reduced fields are obtained, a corresponding solution of the DS equation can be obtained algebraically. The single dromion solution and some kinds of positon solutions are obtained explicitly.

There are a wealth of methods for finding special solutions of a nonlinear partial differential equation (PDE). Some of the most important methods are the inverse scattering transformation (IST) approach [1], the bilinear (BL) method [2], symmetry reductions (SR) [3], Bäcklund and Darboux transformations (BT and DT) [4] etc. In comparison with the linear case, it is known that IST is an extension of the Fourier transformation in the nonlinear case. In addition to the Fourier transformation, there is another powerful tool called the variable separation method in the linear case. However, there is little progress on obtaining some special solutions by means of a corresponding variable separation method in the nonlinear case. Recently, a kind of 'variable separating' procedure has been established by means of symmetry constraints [5, 6]. In this approach, although the independent variables of a reduced field have not totally been separated, the field satisfies some lower-dimensional equations. Each reduced equation does not contain one (or more) independent variable(s) explicitly. For example, in [6], it has been pointed out that for the Kadomtsev–Petviashvili (KP) equation, if *one field* (which depends on three variables, x, y and t) satisfies not only a $(1+1)$-dimensional equation (nonlinear Schrödinger (NLS) equation) for the variables x and y, but also another $(1+1)$-dimensional equation (modified KdV equation) for the variables x and t, then a corresponding solution of the KP equation can be obtained.

Now an interesting question is: Can we find some nontrivial special solutions of an $(n+1)$-dimensional integrable model from *two* reduced fields defined on some subspaces? In this paper we shall present a variable separation procedure for a $(2+1)$-dimensional integrable model, the Davey–Stewartson (DS) equation, by solving its bilinear form and introducing a prior ansatz. It is shown that when *two fields*, $f(x,t)$ and $g(y,t)$, which

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depend only on spacetime (x, t) and (y, t), respectively, satisfy *two* $(1+1)$-dimensional integrable models, the corresponding solutions of the DS equation can be obtained from every pair of solutions of these two field equations.

The DS equation can be written in some variant but equivalent forms. Here we use the same form as in [7]:

$$\mathrm{i}u_t + u_{XX} + u_{YY} - 4u|u|^2 - 2uv = 0 \tag{1}$$

$$v_{XX} - v_{YY} + 4(|u|^2)_{XX} = 0. \tag{2}$$

This system is the shallow water limit of the Benney–Roskes equation [8], where u is the amplitude of a surface wavepacket and v characterizes the mean motion generated by this surface wave. The DS equations system (1) and (2) can also be obtained from the suitable reduction of a self-dual Yang–Mills equation [9].

Introducing new dependent variables F (real) and G (complex) by

$$u = \frac{G}{F} \qquad v = -2\partial_x^2 \log F \tag{3}$$

and rotating the coordinate axes by 45°, equations (1) and (2) can be written as

$$(\mathrm{i}D_t + D_x^2 + D_y^2)G \cdot F = 0 \tag{4}$$

$$D_x D_y F \cdot F = 2|G|^2 \tag{5}$$

where D is the usual bilinear operator [10] defined by

$$D_t^m D_x^n D_y^p F \cdot G \equiv (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n (\partial_y - \partial_{y'})^p F(x, y, t) \cdot G(x', y', t')|_{x=x', y=y', t=t'}. \tag{6}$$

Usually, one uses the bilinear form (4) and (5) to get the multisoliton solution by assuming that F and G are only a sum of several exponentials of $\eta_i = k_i x + l_i y + \omega_i t + \delta_i$. In order to get some different solutions of the DS equation, we look for the solutions of (4) and (5) by means of a variable separation procedure. From the trivial solution of (4) and (5), $G = 0$ and $D_x D_y F \cdot F = 0$, we may get a hint about their nontrivial solution ($G \neq 0$): F can be assumed to possess the following variable separation form,

$$F = f_1(x, t) - g_1(y, t) + c f_1(x, t) g_1(y, t) \tag{7}$$

where $f_1 (= f_1(x, t))$ and $g_1 (= g_1(y, t))$ are y and x independent, respectively, and c is an arbitrary constant. Substituting equation (7) into (5) and writing G also in a variable separation form,

$$G = p(x, t) q(y, t) \exp \mathrm{i}(r(x, t) + s(y, t)) \tag{8}$$

we find that $p = p(x, t)$ and $q = q(y, t)$ should be related to f_1 and g_1 by

$$p = f_{1x}^{\frac{1}{2}} \qquad q = g_{1y}^{\frac{1}{2}}. \tag{9}$$

After substituting equations (7) and (8) with (9) into equation (4) and vanishing both real and imaginary parts, respectively, we obtain

$$g_{1y}^2 (f_{1xx}^2 - 2 f_{1x} f_{1xxx} + 4 f_{1x}^2 (r_t + r_x^2)) + f_{1x}^2 (g_{1yy}^2 - 2 g_{1y} g_{1yyy} + 4 g_{1y}^2 (s_t + s_y^2)) = 0 \tag{10}$$

$$\begin{aligned} &- \left[2(1 + c g_1) g_{1y}^2 f_{1x}^2 - (f_1 - g_1 + c f_1 g_1) f_{1x} g_{1y}^2 \frac{\partial}{\partial x} \right] (f_{1t} + 2 f_{1x} r_x) \\ &+ \left[2(1 - c f_1) f_{1x}^2 g_{1y}^2 + (f_1 - g_1 + c f_1 g_1) f_{1x}^2 g_{1y} \frac{\partial}{\partial y} \right] \\ &\times (g_{1t} + 2 g_{1y} s_y) = 0. \end{aligned} \tag{11}$$

According to the fact that f_1 and r are only functions of x and t and g_1 and s are only functions of y and t, we can obtain the following four equations,

$$f_{1t} + 2f_{1x}r_x + c_1(t)(1 - cf_1) = 0 \tag{12}$$

$$g_{1t} + 2g_{1y}s_y + c_1(t)(1 + cg_1) = 0 \tag{13}$$

$$4r_t f_{1x}^2 + 4f_{1x}^2 r_x^2 + f_{1xx}^2 - 2f_{1x}f_{1xxx} - c_2 f_{1x}^2 = 0 \tag{14}$$

and

$$4s_t g_{1y}^2 + 4g_{1y}^2 s_y^2 + g_{1yy}^2 - 2g_{1y}g_{1yyy} + c_2 g_{1y}^2 = 0 \tag{15}$$

where $c_1 = c_1(t)$ and $c_2 = c_2(t)$ introduced in the variable separation procedure are two arbitrary functions of t.

From equations (12) and (13), we have $(c \neq 0)$

$$r(x,t) = -\int^x \frac{f_t}{2f_x}\,\mathrm{d}x + c_3(t) \tag{16}$$

and

$$s(y,t) = -\int^y \frac{g_t}{2g_y}\,\mathrm{d}y + c_4(t) \tag{17}$$

with

$$f = \left(f_1 - \frac{1}{c}\right)B^{-1} \qquad g = \left(g_1 + \frac{1}{c}\right)B \qquad B = \exp\int^t cc_1(t')\,\mathrm{d}t' \tag{18}$$

and $c_3 = c_3(t)$ and $c_4 = c_4(t)$ being integration functions of time t. Substituting equations (16) and (17) into equations (14) and (15) leads to two $(1+1)$-dimensional equations about fields $f(x,t)$ and $g(y,t)$, respectively:

$$-2f_x^2\int^x \left(\frac{f_t}{f_x}\right)_t \mathrm{d}x + f_x^2(4c_{3t} - c_2) - 2f_x f_{xxx} + f_{xx}^2 + f_t^2 = 0 \tag{19$'$}$$

$$-2g_y^2\int^y \left(\frac{g_t}{g_y}\right)_t \mathrm{d}y + g_y^2(4c_{4t} + c_2) - 2g_y g_{yyy} + g_{yy}^2 + g_t^2 = 0. \tag{20$'$}$$

Dividing equations (19)′ by f_x^2 and differentiating the result equation once with respect to x, we get a trilinear differential equation

$$f_{xx}f_t^2 + f_{tt}f_x^2 - 2f_t f_{xt} f_x + f_{xx}^3 + f_x^2 f_{xxxx} - 2f_x f_{xx} f_{xxx} = 0. \tag{19}$$

In the same way, equation (20)′ can be rewritten as

$$g_{yy}g_t^2 + g_{tt}g_y^2 - 2g_t g_{yt} g_y + g_{yy}^3 + g_y^2 g_{yyyy} - 2g_y g_{yy} g_{yyy} = 0. \tag{20}$$

Actually, equation (20) possesses the same form as equation (19) simply by the transformation $g \to f$, $y \to x$. This symmetry is a natural result of the fact that the DS system (equations (1) and (2) and/or its bilinear form (4) and (5)) is symmetric with respect to space variables x and y. The integrability of equation (19) is guaranteed by the integrability of the DS equation.

Now the important conclusion is that whenever we get any pair of solutions $f(x,t)$ and $g(y,t)$ (one from equation (19) and the other from (20)), we can get a corresponding solution of the DS equation. The space variables x and y have been completely separated to f and g, respectively.

It is necessary to point out that when we get a pair of solutions f and g from (19) and (20), the functions c_3 and c_4 appearing in equations (16) and (17) should be determined by substituting the solutions into equations (19)′ and (20)′ because equations (19)′ and (20)′ are the integral form of equations (19) and (20).

To give out concrete solutions of equations (19) and (20) is still very difficult. Here we write down only some special solutions of equations (19) and (20) (and then the DS equation). The simplest solutions of equations (19) and (20) possess exponential forms

$$\begin{aligned} f &= a_0 + a\exp(kx+\omega t) \equiv a_0 + a\exp\mu_1 \\ g &= b_0 + b\exp(ly+\Omega t) \equiv b_0 + b\exp\mu_2 \end{aligned} \tag{21}$$

with arbitrary constants a_0, b_0, a, b, k, l, ω and Ω. For solutions (21), the integral functions c_3 and c_4 should be fixed as

$$c_3 = \frac{1}{4k^2}\left(k^2\int^t c_2(t')\,\mathrm{d}t' + (k^4-\omega^2)t\right) \qquad c_4 = \frac{1}{4l^2}\left(-l^2\int^t c_2(t')\,\mathrm{d}t' + (l^4-\Omega^2)t\right).$$

Substituting equations (7) and (8) with equations (9), (16)–(18) and (21) into equation (3) we get a so-called dromion solution,

$$\begin{aligned} u = {} & [(akbl)^{\frac{1}{2}}\exp\mathrm{i}\{-\omega x/(2k) - \Omega y/(2l) + c_3 + c_4\}] \\ & \times[\alpha\exp((-\mu_1-\mu_2)/2) + \beta\exp((\mu_1-\mu_2)/2) \\ & +\gamma\exp((\mu_2-\mu_1)/2) + \delta\exp((\mu_1+\mu_2)/2)]^{-1} \end{aligned} \tag{22}$$

with $\alpha = a_0B^{-1} - b_0B + (2/c) + a_0b_0$, $\beta = B^{-1}a(1+b_0B)$, $\gamma = bB(a_0B^{-1}-1)$, $\delta = cab$. Solution (22) decays exponentially in *all* directions after selecting α, β, γ and δ to possess same signs and $abkl > 0$. If the functions c_3 and c_4 are linear in t (i.e. $c_2 =$ constant) and $c_1(t) = 0 (B = 1)$, the dromion solution (22) is just that obtained by other authors using different approaches such as the BT [11], IST [11, 12] and BL direct method [7, 13]. Some arbitrary functions of t (c_2 and $B(t)$) have entered into solution (22). Actually, some arbitrary functions of time t can be included in the dromion solutions because the DS model possesses an infinite-dimensional Lie symmetry group which contains some arbitrary functions of time t [14–16].

It is not very difficult to find out the travelling-wave solutions of equation (19) (and (20)). In addition to the exponential solution (21), one can find four (and only four) other types of solutions:

(i)

$$\begin{aligned} & f = a(kx+\omega t+x_0) + a_0 \qquad g = b(ly+\Omega t+y_0) + b_0 \\ & c_3 = \frac{1}{4k^2}\left(k^2\int^t c_2(t')\,\mathrm{d}t' - \omega^2 t\right) \qquad c_4 = \frac{1}{4l^2}\left(-l^2\int^t c_2(t')\,\mathrm{d}t' - \Omega^2 t\right). \end{aligned} \tag{23}$$

(ii)

$$\begin{aligned} & f = a(kx+\omega t+x_0)^3 + a_0 \qquad g = b(ly+\Omega t+y_0)^3 + b_0 \\ & c_3 = \frac{1}{4k^2}\left(k^2\int^t c_2(t')\,\mathrm{d}t' - \omega^2 t\right) \qquad c_4 = \frac{1}{4l^2}\left(-l^2\int^t c_2(t')\,\mathrm{d}t' - \Omega^2 t\right). \end{aligned} \tag{24}$$

(iii)

$$\begin{aligned} & f = a((kx+\omega t) + \sinh(kx+\omega t+x_0)) + a_0 \\ & g = b((ly+\Omega t) + \sinh(ly+\Omega t+y_0)) + b_0 \\ & c_3 = \frac{1}{4k^2}\left(k^2\int^t c_2(t')\,\mathrm{d}t' + (k^4-\omega^2)t\right) \qquad c_4 = \frac{1}{4l^2}\left(-l^2\int^t c_2(t')\,\mathrm{d}t' + (l^4-\Omega^2)t\right). \end{aligned} \tag{25}$$

(iv)

$$f = a((kx + \omega t) + \sin(kx + \omega t + x_0)) + a_0$$
$$g = b((ly + \Omega t) + \sin(ly + \Omega t + y_0)) + b_0$$
$$c_3 = \frac{1}{4k^2}\left(k^2 \int^t c_2(t')\,\mathrm{d}t' - (k^4 + \omega^2)t\right) \qquad c_4 = \frac{1}{4l^2}\left(-l^2 \int^t c_2(t')\,\mathrm{d}t' - (l^4 + \Omega^2)t\right). \tag{26}$$

The positon solutions (which are locally singular) have been widely investigated in literature for various $(1+1)$-dimensional integrable models like KdV, mKdV, sine–Gordon and Toda-lattice etc [17]. Now we can construct many kinds of positon solutions of the DS equation by selecting any pair of solutions f and g in equations (23)–(26).

To find out more solutions of f (and g) and then the DS equation, one should look for the non-travelling-wave solutions of (19) (and (20)). However, we just give some remarks concerning equation (19) here instead of solving it.

(i) After making a spacetime transformation,

$$\xi = x + \tfrac{1}{2}t \qquad \tau = x - \tfrac{1}{2}t \tag{27}$$

we can write (19) as

$$f_{\xi\xi} f_\tau^2 + f_{\tau\tau} f_\xi^2 - 2 f_\xi f_\tau f_{\xi\tau} + (f_{\xi\xi} + 2 f_{\xi\tau} + f_{\tau\tau})^2 + (f_\tau + f_\xi)^2$$
$$\times (f_{\xi\xi\xi\xi} + 4 f_{\xi\xi\xi\tau} + 6 f_{\xi\xi\tau\tau} + 4 f_{\xi\tau\tau\tau} + f_{\tau\tau\tau\tau}) - 2(f_\xi + f_\tau)$$
$$\times (f_{\xi\xi} + 2 f_{\xi\tau} + f_{\tau\tau})(f_{\xi\xi\xi} + 3 f_{\xi\xi\tau} + 3 f_{\xi\tau\tau} + f_{\tau\tau\tau}) = 0. \tag{28}$$

It is interesting that equation (28) is symmetric with respect to the new 'spacetime' $\{\xi, \tau\}$. Actually, equation (19) will be symmetric for a more generalized transformation, $\xi = x + at$, $\tau = x + bt$. To find some spacetime symmetric integrable models is also an interesting work because of the requirements of relativistic physics.

(ii) Using the following transformations ($\partial_x \partial_x^{-1} = 1$),

$$f_x = \exp \partial_x^{-1} \phi \tag{29}$$
$$f_t = \psi f_x \tag{30}$$

the trilinear equation (19) can be changed to a Boussinesq-type coupled equation system

$$\psi_t = -\phi_{xx} - \phi\phi_x + \psi\psi_x \tag{31a}$$
$$\phi_t = \psi_{xx} + (\psi\phi)_x \tag{31b}$$

where equation (31*b*) is obtained by differentiating equation (30) twice with respect to x. Once the multisoliton solutions of the equation system (31) are obtained, the corresponding multidromion solutions of the DS equation can be obtained by using the transformation relations (29), (30), (16)–(18), (7)–(9) and (3). Substituting equation (21) into (29) and (30) yields

$$\phi = k \qquad \psi = \omega/k. \tag{32}$$

Equation (32) means that the single dromion solution of the DS equation can be obtained from the constant solution of the Boussinesq-type equation (31).

(iii) Though there are some different Boussinesq-type equations, they cannot be changed to the form of (31). One of the most similar equation is the so-called Broer–Kaup system [18],

$$u_t = -u_{xx} + 2uu_x + 2h_x \tag{33a}$$
$$h_t = h_{xx} + 2(hu)_x \tag{33b}$$

which also possesses a trilinear form [19]:

$$\tau(\tau_{xxx}^2 - \tau_{xx}\tau_{xxxx} - \tau_{xt}^2 + \tau_{xx}\tau_{tt}) - \tau_{xx}\tau_t^2 - \tau_{tt}\tau_x^2 + 2\tau_t\tau_{xt}\tau_x + \tau_{xx}^3 + \tau_x^2\tau_{xxxx} - 2\tau_x\tau_{xx}\tau_{xxx} = 0. \tag{34}$$

Equation (34) can also be changed into a 'spacetime' symmetric form by using a similar transformation as equation (27). Obviously, equations (19) and (34) cannot be transformed into each other because of the first term of (34) and the different signs of their t-derivative terms. The problem of how to solve the trilinear equation (19) and/or the Boussinesq-type equation (31) will be studied in a future work.

In summary, using a prior variable separation ansatz to the DS equation, some special solutions of the DS equation can be obtained by means of two $(1 + 1)$-dimensional Boussinesq-type systems which can be written as trilinear forms. The trilinear equations of two $(1+1)$-dimensional fields possess the same form but with different space variables. Combining each solution of one equation with that of the other, we will obtain a corresponding solution of the DS equation. The usual single dromion solution is just the special case in which the solutions of two $(1+1)$-dimensional trilinear equations are all fixed as the exponentials. Many kinds of positon solutions can be found in the $(2+1)$-dimensional integrable DS equation. Finally, we should point out that the variable separation method proposed here can only obtain some special solutions as in other powerful methods. The problem of how to extend the variable separation method to other models and how to get more solutions of the Boussinesq-type equation (31) are worth further studies.

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